

Comment on “Kinetic theory for a mobile impurity in a degenerate Tonks-Girardeau gas”

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In a recent paper [1] Gamayun, Lychkovskiy, and Cheianov studied the dynamics of a mobile impurity weakly coupled to a one-dimensional Tonks-Girardeau gas of strongly interacting bosons. Employing the Boltzmann equation approach, they arrived at the following conclusions: (i) a light impurity, being accelerated by a constant force, F , does *not* exhibit Bloch oscillations, which were predicted and studied in Refs. [2, 3]; (ii) a heavy impurity does undergo Bloch oscillations, accompanied by a drift with the velocity $v_D \propto \sqrt{F}$.

In this comment we argue that the result (i) is an artifact of the classical Boltzmann approximation. The latter misses the formation of the (quasi) *bound-state* between the impurity and a hole. Its dispersion relation $E_b(P, \rho)$ is a *smooth* periodic function of momentum P with the period $2k_F = 2\pi\hbar\rho$, where ρ is a density of the host gas. Being accelerated by a small force, such a bound-state exhibits Bloch oscillations superimposed with the drift velocity $v_D = \mu F$. The mobility μ may be expressed *exactly* [3] in terms of $E_b(P, \rho)$. Result (ii), while not valid at exponentially small forces, indeed reflects an interesting intermediate-force behavior.

The origin of Bloch oscillations is most transparent for a weakly interacting Bose gas, described by the Gross-Pitaevskii (GP) equation. Its solution reveals that a repulsive impurity binds to a dark soliton – a region of depleted host gas. The resulting composite object (the “depleton”) has a periodic dispersion curve and thus exhibits Bloch oscillations, if a sufficiently small force is applied to the impurity. In a strongly interacting Bose gas the GP approach is not applicable, but the bound-state formation still takes place. To illustrate this phenomenon, one may represent the Tonks-Girardeau gas of N hard-core bosons by free fermions created by c_p^\dagger , weakly coupled to a quantum impurity with the coordinate x_i through the density-density interaction ($\hbar = 1$):

$$\hat{H} = -\frac{1}{2m_i} \frac{\partial^2}{\partial x_i^2} + \sum_p \frac{p^2}{2m_h} c_p^\dagger c_p + \gamma \frac{\rho^2}{m_h N} \sum_{p,q} c_p^\dagger c_{p+q} e^{iqx_i} \quad (1)$$

where m_h is the mass of host particles, m_i is the impurity mass and $0 < \gamma \ll 1$ is a dimensionless coupling constant.

Consider a state of the system with total momentum $P > 0$. If $P < P_0 \equiv \min\{m_i v_F, k_F\}$, the low energy states are those where most of the momentum is carried by the impurity. Indeed, the impurity kinetic energy $P^2/(2m_i)$ is less than that of soft particle-hole excitations above the Fermi sea $\sim v_F P$. In the opposite limit $P > P_0$ the low energy states are those where *hole excita-*

tions carry a significant fraction of the entire momentum P . The many-body ground state adiabatically connects between these two limits, signaling strong impurity-hole hybridization at $P > P_0$. Indeed, consider a subspace of the full many-body space, which contains a single hole excitation with momentum $0 < k < 2k_F$ in addition to the impurity with momentum $P - k$ (this restriction is justified in the limit $\gamma \ll 1$). The basis vectors of this subspace are

$$|k; P\rangle = e^{i(P-k)x_i} c_{k_F-k}^\dagger c_{k_F-k} |\text{Fermi Sea}\rangle. \quad (2)$$

The corresponding Schrödinger equation $\sum_{k'} \langle k; P | \hat{H} | k'; P \rangle \psi_P(k') = E \psi_P(k)$ takes the form of the two-particle problem with the *attractive* delta-interaction (formally the attraction arises from anti-commuting the fermionic operators in the last term in Eq. (1)),

$$\left[\frac{(P-k)^2}{2m_i} + E_h(k) \right] \psi_P(k) - \frac{\gamma\rho}{m_h} \int_0^{2k_F} \frac{dk'}{2\pi} \psi_P(k') = E \psi_P(k), \quad (3)$$

where $E_h(k) = v_F k - k^2/(2m_h)$ is the hole kinetic energy (we measure E relative to $NE_F/3 + \gamma\rho^2/m_h$). This problem admits a unique bound-state solution, whose energy $E = E_b(P)$ is found from the integral equation

$$\int_0^{2k_F} \frac{dk'}{\frac{(P-k')^2}{2m_i} + E_h(k') - E_b(P)} = \frac{2\pi m_h}{\gamma\rho}. \quad (4)$$

Its solution represents a non-perturbative correction to the bare impurity dispersion and is completely missed in the Boltzmann equation treatment. We plot $E_b(P)$, along with the continuum of the scattering states, in Fig. 1 for the case of light, $\eta = m_i/m_h < 1$, and heavy, $\eta > 1$ impurity. The gap Δ between the bound-state and the continuum is found to be $\Delta/E_F \sim \gamma^2\eta/(1-\eta)$ for $\eta < 1 - \gamma/\pi^2$ and $P_0 \leq P$, while $\Delta/E_F \sim \exp\{-\pi^2(\eta-1)/\eta\gamma\}$ for $\eta > 1 + \gamma/\pi^2$. For an almost equal mass case $|1-\eta| < \gamma/\pi^2$, one finds $\Delta/E_F \sim \gamma$. We also note that for $\eta = 1$, integrability of Eq. (1) allows access to the exact many-body ground state energy [4] $E_0(P \sim k_F) = E_F - \frac{2\pi^2}{3\gamma} \frac{(P-k_F)^2}{2m_h}$. Remarkably, as one may verify from Eq. (4), $E_b(P \sim k_F) + \gamma\rho^2/m_h = E_0(P \sim k_F)$ for $\gamma \ll 1$, justifying our Hilbert space truncation.

The hard gap between the bound-state and the continuum is an artifact of restricting the particle in Eq. (2) to

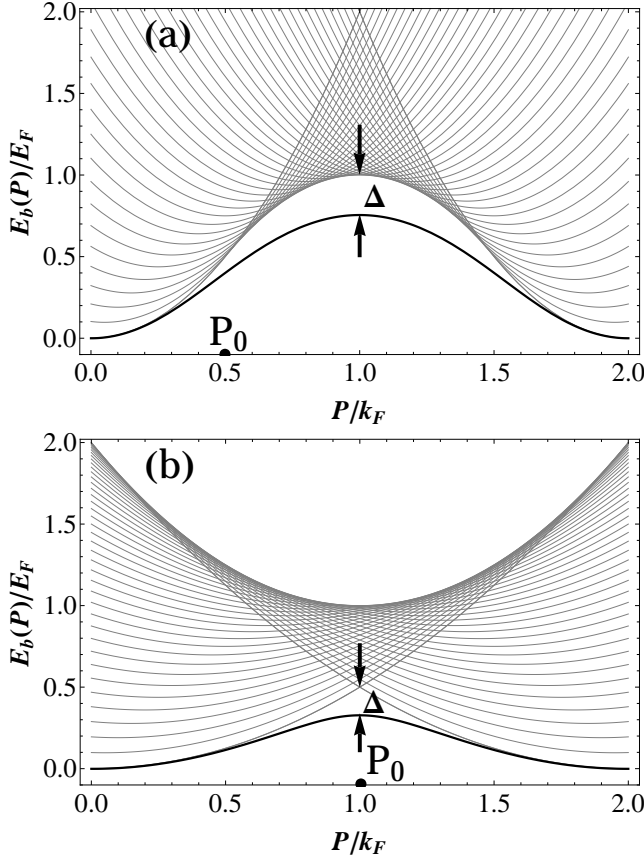


FIG. 1. (Color online) The bound-state $E_b(P)$, Eq. (4), (thick black line) and scattering continuum $\frac{(P-k)^2}{2m_i} + E_h(k)$ for a set of k (thin gray lines) for the light impurity $\eta = 1/2$ (a) and heavy impurity $\eta = 2$ (b). In both cases $\gamma = 0.7$.

be created right at the Fermi momentum k_F . Allowing for slight deviation $c_{k_F}^\dagger \rightarrow c_{k_F+p}^\dagger$, introduces interaction of the bound-state with low energy, $\sim v_F p$, excitations. It is known [4–6] that such interaction transforms the bound-state into the *quasi* bound-state with the power-law (instead of the pole) correlation function. These low energy excitations are responsible for radiation losses and thus for linear mobility μ . They do not, however, destroy the quasi bound-state and associated Bloch oscillations at small applied force.

The Bloch oscillations are destroyed if a large enough force $F > F_{\max}$ is applied to the impurity. The physics of this process is that of the Landau-Zener transition be-

tween the bound-state and the continuum at $P \approx P_0 = k_F \min\{\eta, 1\}$. One may thus estimate the crossover force as $F_{\max} \sim \Delta^2/v$, where $v = v_F \min\{1, 1/\eta\}$. This leads to the following estimate for the maximal force, preserving (nearly) adiabatic bound-state dynamics

$$F_{\max} \propto \frac{k_F^3}{m_h} \begin{cases} \left(\frac{\gamma^2 \eta}{1-\eta}\right)^2, & \eta < 1 - \frac{\gamma}{\pi^2}; \\ \eta \left(\frac{\eta-1}{\eta}\right)^2 e^{-\frac{2\pi^2(\eta-1)}{\gamma\eta}}, & \eta > 1 + \frac{\gamma}{\pi^2}, \end{cases} \quad (5)$$

while for $|1-\eta| < \gamma/\pi^2$, one finds $F_{\max} \propto k_F^3 \gamma^2/m_h$. For $F < F_{\max}$ both light and heavy impurities exhibit Bloch oscillations along with the drift [3], whose velocity scales linearly with the force $v_D = \mu F$.

In Refs. [4, 7] it was shown that for a heavy impurity *away* from the Tonks-Girardeau limit, there exists a phase transition at a critical value of the impurity mass: for $m_i < M_c$ the ground-state is a smooth function of momentum, while for $m_i > M_c$ the ground-state exhibits a cusp singularity at momenta $P = (1+2n)k_F$ for integer n (in the Tonks-Girardeau limit $M_c \rightarrow \infty$). In the latter case the impurity “overshoots” the intersection points at $P = (1+2n)k_F$ and has to emit phonons to reach the ground state. This leads to an enhanced dissipation [7] and thus to super-linear drift velocity

$$v_D \propto F^{1/(1+\alpha)}, \quad (6)$$

where $\alpha \approx 2K - 1$ for $\gamma \ll 1$ and K is the Luttinger parameter of the host.

Notice that in the Tonks-Girardeau limit the validity of the $v_D = \mu F$ response for $\eta > 1$ is limited to an exponentially small force (5). This scale originates from the exponentially narrow region of momenta, where the bound-state exhibits the avoided crossing behavior, Fig. 1(b). For $F > F_{\max}$ the impurity overshoots the avoided crossing and follows the “wrong” parabola before emitting phonons and returning to the ground state. Thus, for $F > F_{\max}$ one may apply Eq. (6) with $K = 1 - \sqrt{F}$, in full agreement with Ref. [1]. An important extension of Ref. [1] is that the super-linear drift (6) is to be expected for moderately heavy impurities $m_h < m_i < M_c$ in an intermediate range of forces where the linear mobility $v_D = \mu F$ is inapplicable.

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[1] O. Gamayun, O. Lyckovskiy and V. Cheianov, arXiv:1402.6362.
[2] D. M. Gangardt and A. Kamenev, Phys. Rev. Lett. **102**, 070402 (2009).
[3] M. Schecter, D. M. Gangardt, and A. Kamenev, Ann. Phys. (N.Y.) **327**, 639 (2011).

[4] A. Lamacraft, Phys. Rev. B **79**, 241105(R) (2009).
[5] A. Kamenev and L. I. Glazman, Phys. Rev. A **80**, 011603(R)(2009).
[6] A. Imambekov, T. L. Schmidt and L. I. Glazman, Rev. Mod. Phys. **84**, 1253 (2012).
[7] M. Schecter, A. Kamenev, D. M. Gangardt and A. Lamacraft, Phys. Rev. Lett. **108**, 207001 (2012).